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Stochastic Inflation and the Lower Multipoles in the CMB Anisotropies

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We generalize the treatment of inflationary perturbations to deal with the non-Markovian colored noise emerging from any realistic approach to stochastic inflation. We provide a calculation of the power-spectrum of the gauge-invariant comoving curvature perturbation to first order in the slow-roll parameters. Properly accounting for the constraint that our local patch of the Universe is homogeneous on scales just above the present Hubble radius, we find a blue tilt of the power-spectrum on the largest observable scales, in agreement with the *WMAP* data which show an unexpected suppression of the low multipoles of the CMB anisotropy. Our explanation of the anomalous behaviour of the lower multipoles of the CMB anisotropies does not invoke any *ad-hoc* introduction of new physical ingredients in the theory.

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I. INTRODUCTION

During the last two decades the inflationary paradigm has become an almost universally accepted scenario to explain the observed large scale flatness and homogeneity of the Universe [1, 2]. At the same time, it provides an efficient mechanism (the stretching of quantum fluctuations up to cosmological scales) to generate those small curvature perturbations (of the order of 10^{-5}) that are believed to produce the observed Cosmic Microwave Background (CMB) temperature anisotropies and to provide the seeds for the formation of the cosmic structures (galaxies and clusters) we observe today.

Stochastic inflation [3] represents one possible approach to the inflationary paradigm, specifically conceived to describe the transition from quantum fluctuations to classical density perturbations [4]. The basic idea is to introduce a cutoff in Fourier space through a suitable time-dependent window function that filters out the modes whose frequency is lower than the comoving horizon size. The inflaton field is thus split in two differently behaving parts: the short-wavelength part has an intrinsically quantum nature, while the coarse-grained one (collecting the remaining super-horizon modes) is treated as classical. A Langevin-like equation of motion for the long-wavelength part is obtained, where the sub-horizon modes enter now as a classical stochastic noise term that perturbs the dynamics of the coarse-grained field.

In most works on this subject [20] the noise is taken to be a *white noise*, whose two-point correlation function is by definition proportional to a Dirac delta function in time. A remarkable feature of stochastic processes involving a white noise, known as Markov property, is that their conditional probability does not depend on times before the constraint. For this particular kind of noise, standard techniques of statistical physics [5] allow to write an evolution equation (the Fokker-Planck equation) for the probability distribution of the inflaton perturbations.

A key issue in the solution of the Langevin or Fokker-Planck equation is the choice of the initial conditions for the perturbations. Many authors [6] agree that it should be consistent to assume the spatial homogeneity of our observable local patch of the Universe, and therefore the vanishing of all fluctuations right before the moment it crosses the horizon size, about 60 e-folds before the end of inflation, since at that time only fluctuations on larger scales could have grown up. Therefore, all points inside the present Hubble radius (at that time contained in the same coarse-graining domain) must have the same local value of the scalar field, although this value can be different from the one assumed in other regions of the Universe. Even if it is generally assumed that inflation started well before the last 60 e-folds, for the white-noise case the evolution of fluctuations is completely insensitive to what happened before that epoch and the constraint really becomes a new initial condition.

However, a white noise arises only if the window function is chosen to be a step-function in Fourier space, therefore if coarse-graining is achieved with the introduc-

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tion of a sharp momentum cutoff. Even though it must be regarded as the mathematical limit of a smoother filtering procedure, it has been pointed out [8] that the step function is a somewhat pathological choice, because it does not satisfy some fundamental properties characterizing "well-behaved" smooth window functions. Conversely, all these window functions always produce a non-divergent colored noise with the same asymptotical behaviour. Unfortunately, the statistical treatment of a colored noise is a rather more difficult matter [9], but in general it is expected that because of their non-Markovianity fluctuations will keep some memory of the evolution before the constraint.

In a previous paper [10], we studied the dynamics of the fluctuations of the coarse-grained inflaton field in a pure de Sitter space-time and found evidence for a blue tilt in the power-spectrum on the largest observable scales as a consequence of the non-Markovian dynamics near the constraint. This is due to the fact that the increased noise correlation time (with respect to the white-noise case) acts as a sort of "inertia" against the growth of the perturbations after the constraint, thereby resulting in a suppression of the power-spectrum on the scales that crossed the horizon in the following few Hubble times.

This is an interesting feature, since the CMB anisotropy measurements made by *WMAP* [11] give some evidence for a suppression of the low multipoles, confirming earlier analogous results found by *COBE* [12]. Although the statistical significance of such a suppression is not large [13], many authors have recently tried to explain this spectral feature either invoking astrophysical effects [14] or introducing some new physical input in the mechanism that generates the perturbations [15].

In this paper, we address the question of the low multipoles suppression from the stochastic inflation point of view, suggesting that this might be simply a consequence of the colored noise. Compared to a white noise, a smooth choice of the window will in fact slightly suppress the contribution to the noise given by the field modes whose frequency is immediately higher than the cutoff scale $\sigma(aH)$ (while enhancing lower frequencies) where σ is a parameter smaller than unity, which is introduced to parametrize to level of arbitrariness – besides the shape of the window function – in the size of the coarse-graining domain. Right after the time τ_* at which we set the homogeneity constraint on our comoving patch of Universe, fluctuations with $k \lesssim \sigma a_* H_*$ will grow less than in the white-noise case before freezing out, and if σ is not too small this suppression can be effective also on observable scales. However, there seems to be no physical motivations to state that the coarse-graining domain has to be very large compared to the horizon size. This assumption indeed appears in early stochastic inflation works, where σ is taken to be much smaller than unity, but this is done essentially for practical reasons, in order to ease up the calculations and to cast away from observable scales the possible effects introduced by the white-noise approximation. This choice is justified *a posteriori* with the argument that the noise correlation function in configuration space is independent of σ up to second order. Nevertheless, a σ dependence still remains (even in the Markovian case) in the power-spectrum.

The plan of the paper is as follows: in Section II we briefly review the formalism of stochastic inflation, extending it to the colored-noise case [10]. In Section III we exactly calculate the colored-noise correlation function for a free massive scalar field in a de Sitter space-time. In Section IV we evaluate the effects of the constraint at τ_* computing the conditional probability distribution of the same scalar field. In Section V we extend our results to the perturbations of the inflaton field (including in our treatment also metric fluctuations) and we compute the power-spectrum $\mathcal{P}_{\mathcal{R}}$ of the gauge-invariant curvature perturbation, taking as the only approximation a first-order slow-roll expansion of the background inflaton field. Finally, our conclusions are drawn in Section VI.

II. EFFECTIVE STOCHASTIC ACTION

We begin our considerations by taking a free scalar field with mass m in a homogeneous and isotropic background space-time, whose metric has the pure Robertson-Walker form

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2 = a^2(\tau)(d\tau^2 - d\mathbf{x}^2), \qquad \text{(II.1)}$$

where $a(t)$ is the scale factor responsible for the expansion of the Universe and τ is the conformal time, defined in such a way that $d\tau = dt/a(t)$.

The ordinary action for a scalar field in a curved space-time with this background metric reads

$$S[\phi] = \int d^4x a^3(t) \frac{1}{2}\left[(\partial_t\phi)^2 - \frac{(\nabla\phi)^2}{a^2} - m^2\phi^2\right], \qquad \text{(II.2)}$$

and the equation of motion for the normal modes $\phi_{\mathbf{k}}$ (assuming a spatially flat Universe these are simply obtained by expanding the field on the 3-dimensional Fourier basis $e^{i\mathbf{k}\cdot\mathbf{x}}/(2\pi)^{3/2}$) is

$$\ddot{\phi}_{\mathbf{k}} + 3H\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\phi_{\mathbf{k}} + m^2\phi_{\mathbf{k}} = 0, \qquad \text{(II.3)}$$

where dots denote ordinary time derivatives.

It is convenient to define a new field variable χ, whose normal modes are related to the Fourier modes $\phi_{\mathbf{k}}$ by

$$\chi_{\mathbf{k}} = \phi_{\mathbf{k}} a(\tau); \qquad \text{(II.4)}$$

in conformal time, indicating with a prime the conformal-time derivative $\partial_\tau = a\partial_t$, the equation of motion for the new field becomes

$$\chi''_{\mathbf{k}} + \left[k^2 + m^2a^2 - \frac{a''}{a}\right]\chi_{\mathbf{k}} = 0, \qquad \text{(II.5)}$$

and can be obtained by variation of the conformal action

$$S[\chi] = -\int d^4x \frac{1}{2}\chi\left[\Box + m^2a^2 - \frac{a''}{a}\right]\chi, \tag{II.6}$$

where the metric determinant does not appear and $\Box = \partial_\tau^2 - \nabla^2$ is the ordinary flat space-time d'Alembert operator.

In the de Sitter case, the Hubble parameter $H \equiv \dot{a}/a$ measuring the expansion rate is strictly constant in time, and the scale factor evolves as

$$a(t) = e^{Ht}, \tag{II.7}$$

while the conformal time, ranging from $-\infty$ to 0, becomes

$$\tau = -\frac{1}{a(t)H}. \tag{II.8}$$

The evolution equation for the $\chi_{\mathbf{k}}$'s can then be recast in the form

$$\chi''_{\mathbf{k}} + \left[k^2 - \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\right]\chi_{\mathbf{k}} = 0, \tag{II.9}$$

where the parameter ν is defined as

$$\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \equiv \frac{3}{2} - \epsilon\,. \tag{II.10}$$

The generic solution to this equation, expressed in terms of Bessel functions of the first and second kind, is

$$c_1\sqrt{|\tau|}J_\nu(k|\tau|) + c_2\sqrt{|\tau|}Y_\nu(k|\tau|); \tag{II.11}$$

requiring each $\chi_{\mathbf{k}}$ to match the plain wave solution $e^{-ik\tau}/\sqrt{2k}$ for $k \gg aH$, when wavelengths are too short to feel any space-time curvature effects, produces the standard Bunch-Davies solution

$$\chi_{\mathbf{k}}(\tau) = \frac{\sqrt{\pi}}{2}\sqrt{|\tau|}H_\nu^{(1)}(k|\tau|), \tag{II.12}$$

where

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x) \tag{II.13}$$

it the Hankel function of the first kind. For the normal modes of the original scalar field ϕ, this gives

$$\phi_{\mathbf{k}}(\tau) = \frac{\sqrt{\pi}}{2}H|\tau|^{3/2}H_\nu^{(1)}(k|\tau|), \tag{II.14}$$

that in the massless case ($\nu = \frac{3}{2}$) becomes

$$\phi_{\mathbf{k}}(\tau) = H\frac{k\tau - i}{\sqrt{2k^3}}e^{-ik\tau}. \tag{II.15}$$

Each mode is said to leave the horizon when the corresponding physical wavelength ak^{-1} encompasses, due to the growth of the scale factor $a(t)$, the size of the cosmological particle horizon H^{-1}. At the horizon crossing we therefore have $|k\tau| = 1$.

It is possible to obtain the sub-horizon part of the field selecting the short-wavelength modes, through a suitable time-dependent high-pass filter in Fourier space. We do this by means of a window function $W_\sigma(k\tau)$ such that $W_\sigma(k\tau) = 0$ for $k|\tau| \ll \sigma$ and $W_\sigma(k\tau) = 1$ for $k|\tau| \gg \sigma$, where the parameter σ (in early stochastic inflation works called ϵ) introduces some level of arbitrariness in the size of the coarse-graining domain, corresponding to the "effective horizon" $\sigma(aH)$. The fine-grained sub-horizon field $\phi_>$ is therefore defined as

$$\phi_> = \int d\mathbf{k}\frac{W_\sigma(k\tau)}{(2\pi)^{3/2}}\left[a_{\mathbf{k}}\phi_{\mathbf{k}}(\tau)e^{-i\mathbf{k}\cdot\mathbf{x}} + h.c.\right], \tag{II.16}$$

while the coarse-grained super-horizon part $\phi_< \equiv \varphi$ is left for the moment unspecified.

An effective equation of motion for the coarse-grained part can simply be obtained by substituting $\phi = \varphi + \phi_>$ directly in the equation of motion (II.3) [3]; all the high-frequency part is then collected into a random noise field which acts as a classical source term for the long-wavelengths part. The quantum fluctuations on sub-horizon scales are averaged into a classical noise term ξ (with a given probability distribution $P[\xi]$) perturbing the super-horizon dynamics: the effects of such a perturbation is then a stochastic process to be studied with the ordinary methods of classical statistical physics. The quantum problem of computing the expectation value of the coarse-grained field φ is thus reduced to the classical problem of evaluating the mean of the solution to the stochastic evolution equation averaged over all possible noise configurations.

However, things are a bit more complicated than this, since we are dealing with true expectation values between the same *in* state, and not simply with *in-out* transition amplitudes (which are the usual objects in ordinary quantum field theory). The calculation of expectation values is a typical problem of out-of-equilibrium field theory [16]. In this framework, an effective action for $\langle\varphi\rangle$ can be obtained via the so-called influence functional method [17], where the splitting of the field is operated in the action, and the short wavelengths are integrated out with a path-integral over all the configurations of the sub-horizon field $\phi_>$. This method actually introduces some extra terms in the effective equation of motion [10]. However, these effects are small and are neglected in the present work.

Since we take the effect of sub-horizon fluctuations on large scales to be small, we can split the field into its statistical mean value φ (satisfying the classical equation of motion (II.3)) and a fluctuation $\delta\varphi[\xi]$, that by definition vanishes when averaged over the ξ's. The stochastic equation of motion for the super-horizon fluctuations can be reduced to

$$\delta\ddot{\varphi}_{\mathbf{k}} + 3H\,\delta\dot{\varphi}_{\mathbf{k}} - \left(\frac{k^2}{a^2} - m^2\right)\delta\varphi_{\mathbf{k}} = \frac{\xi_{\mathbf{k}}}{a^3}, \tag{II.17}$$

or equivalently

$$\delta\chi''_{\mathbf{k}} + \left(k^2 - \frac{a''}{a} + m^2 a^2\right)\delta\chi_{\mathbf{k}} = \xi_{\mathbf{k}}, \tag{II.18}$$

which are the usual equations obtained in stochastic inflation inserting the frequency split of the field directly into the equation of motion. This is a fully classical Langevin-like equation, where the noise ξ (resulting from the average of the quantum fluctuations on sub-horizon scales) stochastically perturbs the super-horizon dynamics, acting as a source for the long-wavelengths field φ, which in turn can be treated as a classical stochastic variable.

The noise ξ is a Gaussian random field, whose configurations are weighted by the functional probability distribution

$$P[\xi] = N \exp\left[-\frac{1}{2}\int d^4x d^4x' \xi(x)\mathbf{A}^{-1}(x,x')\xi(x')\right] \tag{II.19}$$
$$= N \exp\left[-\frac{1}{2}\int d\tau d\tau' d\mathbf{k} d\mathbf{k}' \xi_{\mathbf{k}}(\tau)\mathbf{A}^{-1}_{\mathbf{k},\mathbf{k}'}(\tau,\tau')\xi_{\mathbf{k}}(\tau')\right],$$

where $\mathbf{A}^{-1}_{\mathbf{k},\mathbf{k}'}(\tau,\tau')$ is the functional inverse of

$$\mathbf{A}_{\mathbf{k},\mathbf{k}'}(\tau,\tau') = \delta(\mathbf{k}+\mathbf{k}')\frac{\mathrm{Re}[f(k\tau)f^*(k\tau')]}{2k^3}, \tag{II.20}$$

and

$$f(k\tau) = \sqrt{2k^3}(W''_\sigma \chi_{\mathbf{k}} + 2W'_\sigma \chi'_{\mathbf{k}}). \tag{II.21}$$

This probability distribution allows us to calculate the statistical mean value $\langle \ldots \rangle_S$ of any ξ-dependent quantity averaged over all the noise field configurations, defined as

$$\langle \ldots \rangle_S = \int \mathcal{D}[\xi] \ldots P[\xi]. \tag{II.22}$$

Then, by definition the mean $\langle \xi(\tau)\rangle_S$ of the noise vanishes at all times, while the two-point correlation function, which completely characterizes the statistical properties of the Gaussian noise field, is then by definition

$$\langle \xi_{\mathbf{k}}(\tau)\xi_{\mathbf{k}'}(\tau')\rangle_S = \mathbf{A}_{\mathbf{k},\mathbf{k}'}(t,t') \tag{II.23}$$

In configuration space the correlation function reads

$$\langle \xi(x)\xi(x')\rangle_S = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}\frac{1}{2k^3} \mathrm{Re}[f(k\tau)f^*(k\tau')]. \tag{II.24}$$

The statistical behaviour of the noise thus critically depends on the shape of the filter. Choosing the usual window $W_\sigma(k\tau) = \vartheta(k|\tau| - \sigma)$ we obtain the standard white-noise two-point correlation function. For $\mathbf{x} = \mathbf{x}'$ it reads

$$\langle \xi(x)\xi(x')\rangle_S = \frac{H^3}{4\pi^2}\left(1 + \mathcal{O}(\sigma^2)\right)\delta(t-t'); \tag{II.25}$$

which is highly divergent for $t = t'$ and has a vanishing characteristic correlation time, while a smooth window yields a correlation function with no divergences and a finite correlation time, therefore producing a colored noise. Namely, choosing

$$W_\sigma(k\tau) = 1 - e^{-\frac{k^2\tau^2}{2\sigma^2}}, \tag{II.26}$$

the two-point correlation function for $r = 0$ can be calculated, yielding

$$\langle \xi(t)\xi(t')\rangle_S = \frac{H^4}{8\pi^2}\frac{1}{\cosh^2(H(t-t'))} + \mathcal{O}(\sigma^2), \tag{II.27}$$

that asymptotically behaves like $e^{-2H(t-t')}$. Moreover, it is possible to show that this asymptotic result is quite general for a wide class of smooth window functions [8].

III. FLUCTUATIONS

The particular solution of the evolution equation (II.18) for the fluctuations $\delta\chi_{\mathbf{k}}$ sourced by the noise field ξ can be expressed in terms of the general solutions $\chi_1 = \sqrt{k|\tau|}J_\nu(k|\tau|)$ and $\chi_2 = \sqrt{k|\tau|}Y_\nu(k|\tau|)$ of the homogeneous equation (II.9). This solution reads

$$\delta\chi_{\mathbf{k}}[\xi](\tau) = \int_{\tau_{in}}^{\tau} d\tilde{\tau}\, g(k\tau, k\tilde{\tau})\,\xi_{\mathbf{k}}(\tilde{\tau}), \tag{III.1}$$

where

$$g(k\tau, k\tilde{\tau}) = \frac{\chi_1(k\tau)\chi_2(k\tilde{\tau}) - \chi_2(k\tau)\chi_1(k\tilde{\tau})}{\chi'_1(k\tilde{\tau})\chi_2(k\tilde{\tau}) - \chi'_2(k\tilde{\tau})\chi_1(k\tilde{\tau})} \tag{III.2}$$

and τ_{in} is the beginning of inflation, at which we set the initial condition $\delta\chi_{\mathbf{k}}(\tau_{in}) = 0$.

Keeping this assumption, we now want to introduce in our system the constraint that at a much later time τ_* (roughly about 60 e-folds before the end of inflation) we have no fluctuations in that part of Universe corresponding to the present observable sky. This is motivated by the fact that in our treatment all the points we observe today (over which we measure a substantial homogeneity) were included at τ_* in the same coarse-grained domain. It is thus consistent to assume τ_* the complete homogeneity of the comoving patch of Universe we observe today, all fluctuations on smaller scales being generated later by the noise term. In this approach, we conservatively make no assumptions on the behaviour on larger unobservable scales.

We are thus led to consider (for a given noise configuration) a different solution for the subsequent evolution of the fluctuations, obtained as in (III.1) by starting the integration at τ_*, when a new (stochastic) initial condition holds. In turn, $\delta\chi_{\mathbf{k}}[\xi](\tau_*)$ is determined again from (III.1) with the usual vanishing initial condition at τ_{in}. However, as far as we are dealing with points inside the present observable Universe, we can skip the stochastic

initial conditions τ_* since their inverse Fourier transform is assumed to vanish. Therefore, in configuration space the subsequent evolution of the fluctuations will only contain noise modes integrated after τ_*. That is, for relevant $\mathbf{x}$'s we write

$$\delta\varphi(\mathbf{x},\tau) = \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} \frac{e^{i\mathbf{kx}}}{a} \int_{\tau_*}^{\tau} d\tilde{\tau}\, g(k\tau, k\tilde{\tau})\, \xi_{\mathbf{k}}(\tilde{\tau}) \quad \text{(III.3)}$$

and

$$\delta\varphi(\mathbf{x},\tau_*) = \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} \frac{e^{i\mathbf{kx}}}{a_*} \int_{\tau_{in}}^{\tau} d\tilde{\tau}\, g(k\tau_*, k\tilde{\tau})\, \xi_{\mathbf{k}}(\tilde{\tau}) \ , \quad \text{(III.4)}$$

where the first equation is only valid for scales inside our observed patch of the Universe.

As expected, since the fluctuation $\delta\varphi_{\mathbf{k}}[\xi]$ is linear in ξ, at all times we have that

$$\langle \delta\varphi[\xi](\tau) \rangle_S = 0, \quad \text{(III.5)}$$

while the two-point correlation function in $\mathbf{x}_1$ and $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{r}$ can be obtained integrating the noise correlation function (II.20). We find

$$C(\mathbf{r}) \equiv \langle \delta\varphi[\xi](\mathbf{x}_1,\tau)\delta\varphi[\xi](\mathbf{x}_2,\tau) \rangle_S = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{kr}} \frac{|I_1(k)|^2}{2k^3}, \quad \text{(III.6)}$$

where

$$I_1(k) = \frac{\sqrt{2k^3}}{a} \int_{\tau_*}^{\tau} d\tilde{\tau}\, g(k\tau, k\tilde{\tau})(W_\sigma''\chi_{\mathbf{k}} + 2W_\sigma'\chi_{\mathbf{k}}'). \quad \text{(III.7)}$$

In order to evaluate this integral, we first change the integration variable from $\tilde{\tau}$ to $x = k\tilde{\tau}$ and apply the relation

$$J_\nu(x)\frac{d}{dx}Y_\nu(x) - \frac{d}{dx}J_\nu(x)Y_\nu(x) = \frac{2}{\pi x} \quad \text{(III.8)}$$

in the denominator of $g(k\tau, k\tilde{\tau})$. Second, we integrate by parts the W_σ'' term and and use again (III.8). The integral can thus be written as

$$I_1(k) = H\sqrt{\frac{\pi k^3|\tau|^3}{2}} \left[\frac{\pi}{2} x\, W_\sigma'(x) \Big[Y_\nu(k|\tau|)J_\nu(x) - J_\nu(k|\tau|)Y_\nu(x) \Big] H_\nu(x) \Big|_{k|\tau|}^{k|\tau_*|} + H_\nu(k|\tau|) \int_{k|\tau|}^{k|\tau_*|} dx\, \frac{d}{dx} W_\sigma(x) \right], \quad \text{(III.9)}$$

containing only a boundary term and a trivial integration.

In the same way we can calculate the correlation function evaluated at τ_*, yielding

$$C_*(\mathbf{r}) \equiv \langle \delta\varphi[\xi](\mathbf{x}_1,\tau_*)\delta\varphi[\xi](\mathbf{x}_2,\tau_*) \rangle_S = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{kr}} \frac{|I_2(k)|^2}{2k^3} \quad \text{(III.10)}$$

where $I_2(k)$ has the same form as $I_1(k)$ but it refers to the time interval $[\tau_{in}, \tau_*]$. However, the boundary term is now proportional to $W_\sigma'(k|\tau_{in}|)$ and its contribution to the correlation function will be effective only on the scales that crossed the horizon at the beginning of inflation. Therefore, since in most realistic models $\tau_* \gg \tau_{in}$, the boundary term in our treatment is completely negligible, and we have

$$I_2(k) = H\sqrt{\frac{\pi k^3|\tau_*|^3}{2}} H_\nu(k|\tau_*|) \Big(W_\sigma(k|\tau_{in}|) - W_\sigma(k|\tau_*|) \Big). \quad \text{(III.11)}$$

Moreover, we can also define the (generally non-vanishing) mixed correlation function

$$M(\mathbf{r}) \equiv \langle \delta\varphi[\xi](\mathbf{x}_1,\tau)\delta\varphi[\xi](\mathbf{x}_2,\tau_*) \rangle_S = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{kr}} \frac{\mathrm{Re}[I_1(k)I_2^*(k)]}{2k^3}, \quad \text{(III.12)}$$

involving the scalar field perturbations evaluated at different times.

IV. PROBABILITY DISTRIBUTION

We can construct the joint probability distribution $P\big[\delta\varphi_1, \delta\varphi_2, c\big]$ that the stochastic variables $\delta\varphi[\xi](\mathbf{x}_1)$ and $\delta\varphi[\xi](\mathbf{x}_2)$ satisfy the constraint c (i.e. they vanish) at time τ_* *and* assume the values $\delta\varphi_1$ and $\delta\varphi_2$ at time τ by taking the statistical average

$$P\big[\delta\varphi_1,\delta\varphi_2,c] = \left\langle \prod_{i=1,2} \delta\Big(\delta\varphi_i - \delta\varphi[\xi](\mathbf{x}_i,\tau)\Big)\delta\Big(\delta\varphi[\xi](\mathbf{x}_i,\tau_*)\Big)\right\rangle_S \tag{IV.1}$$
$$= \int \frac{d\alpha_1 d\alpha_2 d\beta_1 d\beta_2}{(2\pi)^4} e^{-i\sum \alpha_i\delta\varphi_i} \int \mathcal{D}[\xi] e^{i\sum \alpha_i\delta\varphi[\xi](\mathbf{x}_i,\tau)+i\sum\beta_i\delta\varphi[\xi](\mathbf{x}_i,\tau_*)} P[\xi].$$

Since the field $\delta\varphi[\xi]$ is linear in the noise ξ, the evaluation of the mean consists in a simple Gaussian functional integration yielding (repeated indices are summed)

$$P\big(\delta\varphi_1,\delta\varphi_2,c\big) = \int \frac{d\alpha_1 d\alpha_2}{(2\pi)^2} e^{-i\sum \alpha_i\delta\varphi_i - \frac{1}{2}\alpha_i C(\mathbf{r}_{ij})\alpha_j} \int \frac{d\beta_1 d\beta_2}{(2\pi)^2} e^{-\frac{1}{2}\beta_i C_*(\mathbf{r}_{ij})\beta_j - \alpha_i M(\mathbf{r}_{ij})\beta_j}, \tag{IV.2}$$

where $\mathbf{r}_{11} = \mathbf{r}_{22} = 0$ while $\mathbf{r}_{12} = \mathbf{r}_{21} = \mathbf{r}$, and thus the result now only involves 2×2 symmetric correlation matrices.

According to Bayes theorem, the conditional probability $P_c\big(\delta\varphi_1,\delta\varphi_2\big)$ is the joint probability $P\big(\delta\varphi_1,\delta\varphi_2,0,0\big)$ normalized by the probability $P_*(0,0)$ of the constraint. The latter can be evaluated exactly in the same way, taking the mean value over the noise configurations of the product of two δ-functions constraining the value of the fluctuations at τ_*. Following the same steps as before, the constraint probability reads

$$P_*\big(c\big) = \int \frac{d\beta_1 d\beta_2}{(2\pi)^2} e^{-\frac{1}{2}\beta_i C_*(\mathbf{r}_{ij})\beta_j}, \tag{IV.3}$$

and thus provides the correct normalization needed in order to evaluate in (IV.2) the Gaussian integration over β_1 and β_2. With these results, we are now able to compute the conditional two-point correlation function

$$\langle\delta\varphi(\mathbf{x}_1)\delta\varphi(\mathbf{x}_2)\rangle_c \equiv \int \delta\varphi_1\delta\varphi_2 \frac{P\big(\delta\varphi_1,\delta\varphi_2,c\big)}{P_*(c)}$$
$$= \int \delta\varphi_1\delta\varphi_2 \int \frac{d\alpha_1 d\alpha_2}{(2\pi)^2} e^{-i\sum\alpha_i\delta\varphi_i - \frac{1}{2}\alpha_i\big(C(\mathbf{r}_{ij}) - M(\mathbf{r}_{ik})[C_*(\mathbf{r}_{kl})]^{-1}M(\mathbf{r}_{lj})\big)\alpha_j}$$
$$= C(\mathbf{r}) - 2\frac{M(0)M(\mathbf{r})}{C_*(0)+C_*(\mathbf{r})} + \frac{C_*(0)\,(M(0)-M(\mathbf{r}))^2}{C_*^2(0)-C_*^2(\mathbf{r})}. \tag{IV.4}$$

This result looks rather messy, but it considerably simplifies if we take the very reasonable limit $\tau_* \gg \tau_{in}$ (as it is the case in most inflationary models). Actually, we can assume that $I_2(k)$ is given by (III.9) (with τ_* instead of τ) neglecting the boundary term, since this would modify the correlation function only on extremely large scales (those that crossed the horizon at the beginning of inflation). We then get

$$\frac{|I_2(k)|}{Ha_*} \sim (k|\tau_*|)^\epsilon\big(W_\sigma(k|\tau_{in}|) - W_\sigma(k|\tau_*|)\big), \tag{IV.5}$$

which is non-vanishing only for $|\tau_{in}|^{-1} \lesssim k \lesssim |\tau_*|^{-1}$, and since the spatial fluctuations are effective for $k \gtrsim r^{-1} \gtrsim |\tau_*|^{-1}$ and are therefore damped by the cutoff introduced by $W_\sigma(k|\tau_*|)$ we roughly have

$$C_*(\mathbf{r}) \sim \frac{H^2}{4\pi^2}\int_{|\tau_{in}|^{-1}}^{|\tau_*|^{-1}} \frac{dk}{k}(k|\tau_*|)^{2\epsilon}, \tag{IV.6}$$

which is divergent for $|\tau_{in}| \to \infty$ and $\epsilon \to 0$.

However, the dependence on τ_{in} contained in $I_2(k)$ is rapidly saturated in the mixed correlation function: actually, it enters in $M(\mathbf{r})$ only through the product $I_1(k)I_2^*(k)$, and $I_1(k)$ (which is again obtained from (III.9) with τ_* instead of τ_{in}) significantly differs from 0 only for $k \gtrsim |\tau_*|^{-1}$. Therefore, when integrating over $\frac{dk}{k}$ the modes responsible for the τ_{in}-dependence (*i.e.* those such that $k \sim |\tau_{in}|^{-1}$) are completely damped, $M(\mathbf{r})$ is constant in the limit $\tau_{in} \ll \tau_*$, and in this approximation we get

$$\langle\delta\varphi(\mathbf{x}_1)\delta\varphi(\mathbf{x}_2)\rangle_c \simeq C(\mathbf{r}). \tag{IV.7}$$

The power-spectrum $\mathcal{P}_{\delta\varphi}(k)$ of the fluctuations, defined so that

$$\langle\delta\varphi(\mathbf{x}_1)\delta\varphi(\mathbf{x}_2)\rangle_c = \frac{1}{4\pi}\int d\mathbf{k}\, e^{i\mathbf{k}\mathbf{r}}\frac{\mathcal{P}_{\delta\varphi}(k)}{k^3}, \tag{IV.8}$$

becomes

$$\mathcal{P}_{\delta\varphi}(k) = \frac{1}{4\pi^2}|I_1(k)|^2. \tag{IV.9}$$

For relevant scales and small values of σ we can neglect the boundary term in (III.9). In the small-σ limit we therefore recover the standard scale-invariant result $\mathcal{P}_{\delta\varphi}(k) = H^2/4\pi^2$.

V. CURVATURE PERTURBATIONS

We have so far treated the scalar field perturbations generated during a de Sitter stage, when the evolution of the scale factor is *a priori* fixed independently of the behaviour of the scalar field. However, if this scalar field is the inflaton, it is the dominating component driving the accelerated expansion; its perturbations will then in turn modify the energy-momentum tensor, inducing fluctuations in the metric and specially in the scale factor evolution, via the slow-roll Friedmann equation $H^2 \simeq (8\pi G/3)V(\phi)$. Besides scalar field perturbations $\delta\varphi$, we must now also consider the small metric perturbations. Both the metric fluctuations and the inflaton perturbation $\delta\varphi$ will assume different values depending on the choice of the coordinate frame. Therefore, in order to avoid this ambiguity and deal only with physical degrees of freedom, it is convenient to define the gauge-invariant comoving curvature perturbation $\mathcal{R} = \psi + H(\delta\varphi/\dot{\phi})$ which measures the intrinsic spatial curvature on hypersurfaces of constant time [2]. Defining in conformal time $z = a^2\phi'/a'$ and $u = -z\,\mathcal{R}$, the latter variable satisfies the equation of motion

$$u'' - \nabla^2 u - \frac{z''}{z}u = 0. \tag{V.1}$$

Expanding the last term to first order in the slow-roll parameters $\epsilon = -\dot{H}/H^2$ and $\eta = V''/3H^2$, one finds

$$\frac{z''}{z} \simeq \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right), \tag{V.2}$$

where $\nu \simeq \frac{3}{2} + 3\epsilon - \eta$.

Therefore, in the slow-roll approximation the gauge-invariant normal modes $u_{\mathbf{k}}$ satisfy the same equation of motion (II.9), the only difference being in the definition of the parameter ν labelling the solutions. We can thus apply also to u the stochastic formalism we developed in Sec. II for the case of a massive scalar field in a pure de Sitter background, obtaining

$$u''_{\mathbf{k}} + \left[k^2 - \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\right]u_{\mathbf{k}} = \xi_{\mathbf{k}}. \tag{V.3}$$

We can then apply to $\mathcal{R}$ the results derived for the power-spectrum of the perturbations of a test scalar field, concluding that also for the curvature perturbation we have $\mathcal{P}_{\mathcal{R}}(k) \propto |I_1(k)|^2$.

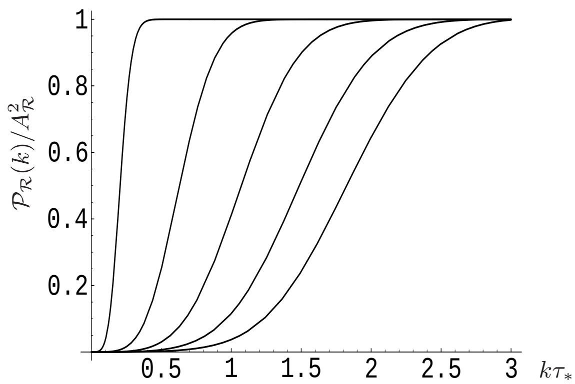

Figure 1: Conditional power-spectrum obtained with the Gaussian window (II.26) in the limit $\tau_*/\tau_{in} \simeq \infty$, for different values of σ varying from 0.1 (left curve) to 0.9 (right curve).

More precisely, recalling (III.9) and taking the limit $k|\tau| \ll \sigma \lesssim 1$ (which is reasonably satisfied on cosmological scales) we get

$$\mathcal{P}_{\mathcal{R}}(k) = A^2_{\mathcal{R}}\Big|\frac{\pi}{2}k|\tau_*|\, J_\nu(k|\tau_*|)H_\nu(k|\tau_*|)W'_\sigma(k|\tau_*|) + iW_\sigma(k|\tau_*|)\Big|^2 (k|\tau|)^{2\eta-6\epsilon}. \tag{V.4}$$

If the boundary term in (III.9) can be neglected, the power-spectrum becomes

$$\mathcal{P}_{\mathcal{R}}(k) = A^2_{\mathcal{R}}W^2_\sigma(k|\tau_*|)(k|\tau|)^{2\eta-6\epsilon}. \tag{V.5}$$

thereby showing, since $W^2_\sigma < 1$, the presence of a blue-tilt on the largest observable scales. However, in the limit where $\sigma \ll k|\tau_*|$ (since $W_\sigma(k|\tau_*|) \simeq 1$) we recover the ordinary result

$$\mathcal{P}_{\mathcal{R}}(k) = A^2_{\mathcal{R}}(k|\tau|)^{2\eta-6\epsilon}. \tag{V.6}$$

For generic values of σ, we show in Fig. 1 the power-spectrum obtained with the Gaussian window function (II.26), where the only approximation is that of considering $\tau_{in} \ll \tau_*$. For this particular choice of the filter, we observe a blue tilt of the power-spectrum for $k \lesssim 3\sigma a_* H_*$.

This blue tilt stems from the fact that a smooth window function does not make a sharp separation in Fourier space but it gradually weighs the modes, allowing for a small low-frequency contribution to the fine-grained part of the field (in term of which the noise is defined) while depleting modes whose wavelength is immediately smaller than the cutoff scale. The colored noise originated from such a window is thus able to generate fewer fluctuations than a white noise on scales slightly smaller than the comoving coarse-graining domain.

As a consequence, since we impose the constraint that in our comoving patch of Universe the fluctuations can

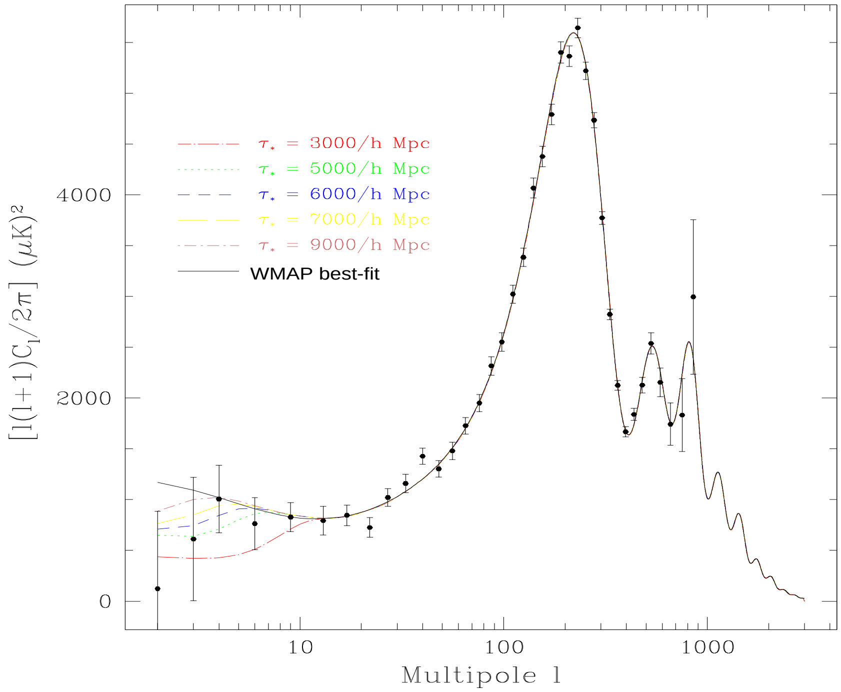

Figure 2: . Angular power spectra of the CMB temperature anisotropies obtained with the Gaussian window (II.26) and different values of the constraint time τ_*. The cosmological parameters used in the computation of the radiation transfer functions are $\Omega_b = 0.046$, $\Omega_{CDM} = 0.224$, $\Omega_\Lambda = 0.730$ and $h = 0.72$, corresponding to the best fit of the WMAP data [19]. Dots represent the WMAP Binned Combined TT Power Spectrum, obtained from the LAMBDA website: http://lambda.gsfc.nasa.gov/.

grow only after τ_*, the scales that are leaving the horizon in the following few Hubble times receive less "random kicks" before freezing out than in the white-noise case. Therefore, the power-spectrum is a function of k smoothly interpolating between the values 0 and 1 it assumes for small and large k, respectively.

This power-spectrum can be used to calculate the CMB multipoles predicted by a specific choice of the window function W. Quite generally, we will expect to find a suppression of the lowest multipole, which are sensitive to a modification of the power-spectrum on this very large scales. However, in order to quantify this suppression one needs to choose the shape of the window function and the precise time τ_* at which the constraint is set. For the Gaussian window (II.26), the predicted CMB multipoles are plotted in Fig. 2 for different values of τ_*. The cosmological parameter used to compute (with CMBfast [18]) the radiation transfer functions are the best fit of the *WMAP* data [11]. The significance of the low multipoles suppression varies depending on the choice of the constraint time: the effect is strong for $\tau_* \simeq 3000h^{-1}$ Mpc (corresponding to the present horizon size), while it becomes fainter for earlier values of τ_* and it is practically absent for $\tau_* \simeq 9000h^{-1}$ Mpc.

VI. CONCLUSIONS

In this paper we calculated the power-spectrum of the comoving curvature perturbation $\mathcal{R}$ in the framework of stochastic inflation with colored noise. We can conclude that a careful analysis of the stochastic behaviour of the fluctuations, for a physically plausible choice of the window function W_σ (therefore excluding the step function), leads to a curvature power-spectrum showing a relevant blue tilt for scales such that $k \sim \sigma a_* H_*$, corresponding to physical lengths about σ^{-1} times greater than the present Hubble radius.

The possibility of finding a blue tilt on observable scales will then depend on the size of the coarse-graining domain at which thehomogeneity constraint is set. If $\sigma a_* H_*$ is of the order of $a_0 H_0$ (today's Hubble radius), then also the biggest scales inour observable Universe get tilted, while if it is much smaller this effect might be completely unobservable. We computed the predicted angular power-spectrum C_l's of CMB temperature anisotropies for a particular choice of the colored-noise window function, finding a suppression of the lowest multipoles whose importance depends on the precise value of τ_*: this effect actally becomes more and more evident as τ_* approaches $3000h^{-1}$ Mpc (the size of the present Hubble radius), whike it is practically absent for $\tau_* \simeq 9000h^{-1}$ Mpc. This is an interesting

feature because the recent measurements of the CMB anisotropy performed by *WMAP* seem to give evidence of a lack of power on the largest observed scales with respect to the predictions of the standard inflationary scenario [19]. In our approach, this deviation from the standard result of an almost scale-invariant perturbation spectrum is achieved without any new physical input in the theory, being merely a consequence of the choice of initial conditions once some over-simplifying approximations (namely, the white-noise choice) are removed. We find it encouraging that no introduction of new physical ingredients is needed to explain the anomalous behaviour of the lower multipoles of CMB anisotropies.

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